Entanglement Content of Quasiparticle Excitations

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(Received 19 June 2018; revised manuscript received 20 September 2018; published 26 October 2018)

We investigate the quantum entanglement content of quasiparticle excitations in extended many-body systems. We show that such excitations give an additive contribution to the bipartite von Neumann and Rényi entanglement entropies that takes a simple, universal form. It is largely independent of the momenta and masses of the excitations and of the geometry, dimension, and connectedness of the entanglement region. The result has a natural quantum information theoretic interpretation as the entanglement of a state where each quasiparticle is associated with two qubits representing their presence within and without the entanglement region, taking into account quantum (in)distinguishability. This applies to any excited state composed of finite numbers of quasiparticles with finite de Broglie wavelengths or finite intrinsic correlation length. This includes particle excitations in massive quantum field theory and gapped lattice systems, and certain highly excited states in conformal field theory and gapless models. We derive this result analytically in one-dimensional massive bosonic and fermionic free field theories and for simple setups in higher dimensions. We provide numerical evidence for the harmonic chain and the two-dimensional harmonic lattice in all regimes where the conditions above apply. Finally, we provide supporting calculations for integrable spin chain models and other interacting cases without particle production. Our results point to new possibilities for creating entangled states using many-body quantum systems.

DOI: 10.1103/PhysRevLett.121.170602

Introduction.—Measures of entanglement, such as entanglement entropy (EE) [1] and entanglement negativity [2–7], have attracted much attention in recent years, both theoretically [8–10] and experimentally [11,12]. Quantum entanglement encodes correlations between degrees of freedom (d.o.f.) associated with independent factors of the Hilbert space, and as such, it separates quantum correlations from the particularities of observables. As a consequence, the entanglement in extended systems encodes, in a natural fashion, universal properties of the state. For instance, at criticality, the entanglement of ground states provides an efficient measure of universal properties of quantum phase transitions, such as the (effective) central change of the corresponding conformal field theory (CFT) and the primary operator content [13–22]. Near criticality, it is universally controlled by the masses of excitations [23–25]. In states that are highly excited, with finite energy densities, the entanglement gives rise to local thermalization effects: at the heart of the eigenstate thermalization hypothesis [26–30], the large entanglement between local d.o.f. and the rest of the system effectively generates a Gibbs ensemble (or generalized Gibbs ensemble in integrable systems). The entanglement effects of a finite number of excitations are less known. Some results are available in critical systems: using the methods of Holzhey, Larsen, and Wilczek [14], combining a geometric description with Riemann uniformization techniques in CFT, it was shown in [31,32] that certain excitations, with energies tending to zero in the large volume limit, correct the ground state entanglement by power laws in the ratio of length scales. Various few-particle states have also been studied in special cases of integrable spin chains [33–37].

In this Letter, we propose a universal formula with a simple quantum information theoretic interpretation for the entanglement content of states with quasiparticle excitations. We consider the von Neumann and Rényi EEs: these are measures of the amount of quantum entanglement, in pure quantum states, between the d.o.f. associated with two sets of independent observables whose union is complete on the Hilbert space. We use the setup where the Hilbert space is factorized as $\mathcal{H}_A \otimes \mathcal{H}_B$, according to two complementary spatial regions A and B, of typical length scales (diameters) ℓ_A and ℓ_B, respectively. The regions can be of generic geometry and connectedness. Let ξ be the correlation length, and $\zeta = \max\{2\pi/|\vec{p}_i|\}$, where $\vec{p}_i$ are the momenta, be the maximal particles' de Broglie wavelength. We find exact formulas in the limit $\min(\xi,\zeta) \ll \min(\ell_A,\ell_B)$, independent of the model studied, of the connectedness or shape of the entanglement region, and of the dimension. For instance, this condition includes the limit of large regions' diameters $\ell_A, \ell_B \to \infty$ in massive quantum field theory (QFT) and gapped quantum lattice systems, where the correlation length ξ is finite. It also includes this same limit in certain states of conformal field

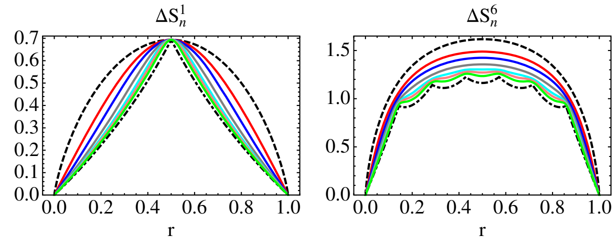

FIG. 1. The functions (3) and (8) for $k = 1, 6$ and $n = 2, 3, 5, 8, 11, 17$ and the limits $n \to 1$ (von Neumann) and $n \to \infty$ (single-copy). The outer-most curve is the von Neumann entropy and the inner-most curve is the single-copy entropy.

theory and gapless models whose energies are finitely separated from that of the ground state, and which have well-defined particle content with finite momenta (finite de Broglie's wavelengths). The results extend the "semiclassical" form discussed in the context of spin chains in [33]. They have a very natural qubit interpretation where qubits representing the particles are entangled according to the particles' distribution in space, taking into account quantum indistinguishability in the bosonic case. Quasiparticle excitations are ubiquitous in many-body quantum systems, and our results are expected to apply to a large family of states with well defined quasiparticle content. We give evidence by studying a variety of clear-cut cases of different dimensions.

Results.—Consider a bipartition of a system $C = A \cup B$ in a state $|\Psi\rangle$ composed of a number k of quasiparticles. In infinite volume, the notion of quasiparticles is natural via the theory of scattering states [38,39], and in finite but large volumes, there exist corresponding excited states, defined unambiguously up to exponentially decaying corrections in the volume. Let the reduced density matrix associated with subsystem A be $\rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$. The Rényi EE is the Rényi entropy of this reduced density matrix

$$S_n^\Psi(A,B) = \frac{\log \mathrm{Tr}\rho_A^n}{1-n}. \tag{1}$$

From (1) we may compute the von Neumann EE as $S_1^\Psi(A,B) \coloneqq \lim_{n\to 1} S_n^\Psi(A,B)$ and the so-called single-copy entropy [40–42] as $S_\infty^\Psi(A,B) \coloneqq \lim_{n\to\infty} S_n^\Psi(A,B)$. For large system size and fixed entanglement region, one expects the entanglement entropies to tend to those of the ground state. Therefore, we concentrate on the nontrivial limit where both the full system C and the entanglement region A are large, scaled simultaneously, $A \mapsto \lambda A$ and $B \mapsto \lambda B$. Let $r = \mathrm{Vol}_d(A)/\mathrm{Vol}_d(C)$ be the ratio of the d-dimensional hypervolume of the region to that of the system. We compute the difference $\Delta S_n^\Psi(A,B) = S_n^\Psi(A,B) - S_n^0(A,B)$ between the Rényi entropy in the excited state $|\Psi\rangle$ and in the ground (vacuum) state $|0\rangle$, in this limit,

$$\Delta S_n^\Psi(r) \coloneqq \lim_{\lambda\to\infty} \Delta S_n^\Psi(\lambda A, \lambda B). \tag{2}$$

This is the contribution of the excitations to the entanglement, or "excess entanglement" as named in [31,32].

We find that, for a wide variety of quantum systems, the results depend only on the proportion r of the system's volume occupied by the entanglement region and are largely independent of the momenta of the quasiparticles. Suppose the state is formed of k particles of equal momenta. Denoting $\Delta S_n^\Psi(r) = \Delta S_n^k(r)$, we find

$$\Delta S_n^k(r) = \frac{\log \sum_{q=0}^k f_q^k(r)^n}{1-n},$$
$$\Delta S_1^k(r) = -\sum_{q=0}^k f_q^k(r) \log f_q^k(r), \tag{3}$$

with $f_q^k(r) = \binom{k}{q} r^q (1-r)^{k-q}$. For a state composed of k particles divided into groups of k_i particles of equal momenta $\vec{p}_i$, with $i = 1, 2, \ldots$, and $\sum_i k_i = k$, we denote $\Delta S_n^\Psi(r) = \Delta S_n^{k_1,k_2,\cdots}(r)$ and have

$$\Delta S_n^{k_1,k_2,\cdots}(r) = \sum_i \Delta S_n^{k_i}(r). \tag{4}$$

In particular, for k particles of distinct momenta, the result is k times that for a single particle, which is

$$\Delta S_n^1(r) = \frac{\log(r^n + (1-r)^n)}{1-n},$$
$$\Delta S_1^1(r) = -r\log r - (1-r)\log(1-r). \tag{5}$$

We observe that, in all cases, the entanglement is maximal at $r = 1/2$. For k distinct-momenta particles, the maximum is $k \log 2$, while when some particles have coinciding momenta, the maximal value is smaller. Interestingly, single-copy entropies present nonanalytic features. For distinct momenta, we have

$$\Delta S_\infty^1(r) = \begin{cases} -\log(1-r) & \text{for } 0 \le r < \frac{1}{2} \\ -\log r & \text{for } \frac{1}{2} \le r \le 1. \end{cases} \tag{6}$$

Again, the result is just multiplied by k for a state consisting of k distinct-momentum particles. For equal momenta, it is a function which is nondifferentiable at k points in the interval $r \in (0,1)$ [generalizing (6)]. The positions of these cusps are given by the values

$$r = \frac{1+q}{1+k} \quad \text{for } q = 0, \ldots, k-1, \tag{7}$$

and the single copy entropy is given by

$$\Delta S_\infty^k(r) = -\log f_q^k(r) \quad \text{for } \frac{q}{1+k} \le r < \frac{1+q}{1+k}, \tag{8}$$

and $q = 0, \ldots, k$. See Fig. 1 for a numerical evaluation.

The results take their full meaning under a quantum information theoretic interpretation that combines a "semi-classical" picture of particles with quantum indistinguishability. Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_{\rm int} \otimes \mathcal{H}_{\rm ext}$. Each factor $\mathcal{H}_{\rm int} \simeq \mathcal{H}_{\rm ext}$ is a tensor product $\otimes_i \mathcal{H}^{k_i}$ of Hilbert spaces $\mathcal{H}^{k_i} \simeq \mathbb{C}^{k_i+1}$ for k_i indistinguishable qubits, with, as above, $\sum_i k_i = k$. We associate $\mathcal{H}_{\rm int}$ with the interior of the region A and $\mathcal{H}_{\rm ext}$ with its exterior, and we identify the qubit state 1 with the presence of a particle and 0 with its absence. We construct the state $|\Psi_{\rm qb}\rangle \in \mathcal{H}$ under the picture according to which equal-momenta particles are indistinguishable, and a particle can lie anywhere in the full volume of the system with flat probability. That is, any given particle has probability r of lying within A, and $1-r$ of lying outside of it. We make a linear combination of qubit states following this picture, with coefficients that are square roots of the total probability of a given qubit configuration, taking proper care of (in)distinguishability. For instance, for a single particle,

$$|\Psi_{\rm qb}\rangle = \sqrt{r}|1\rangle \otimes |0\rangle + \sqrt{1-r}|0\rangle \otimes |1\rangle, \qquad (9)$$

as either the particle is in the region, with probability r, or outside of it, with probability $1-r$. If two particles of coinciding momenta are present, then we have

$$|\Psi_{\rm qb}\rangle = \sqrt{r^2}|2\rangle \otimes |0\rangle + \sqrt{2r(1-r)}|1\rangle \otimes |1\rangle + \sqrt{(1-r)^2}|0\rangle \otimes |2\rangle, \qquad (10)$$

as either the two particles are in the region, with probability r^2, or one is in the region and one outside of it (no matter which one), with probability $2r(1-r)$, or both are outside the region, with probability $(1-r)^2$. For two particles of different momenta,

$$|\Psi_{\rm qb}\rangle = \sqrt{r^2}|11\rangle \otimes |00\rangle + \sqrt{(1-r)^2}|00\rangle \otimes |11\rangle + \sqrt{r(1-r)}(|10\rangle \otimes |01\rangle + |01\rangle \otimes |10\rangle), \qquad (11)$$

counting the various ways two distinct particles can be distributed inside or outside the region. Higher-particle states can be constructed similarly. The results stated above are then equivalent to the identification $\Delta S_n^{\Psi}(r) = S_n^{\Psi_{\rm qb}}(r)$, where

$$S_n^{\Psi_{\rm qb}}(r) = \frac{\log {\rm Tr}\rho_{\rm int}^n}{1-n}, \qquad \rho_{\rm int} = {\rm Tr}_{\rm ext}|\Psi_{\rm qb}\rangle\langle\Psi_{\rm qb}|. \qquad (12)$$

Methods.—The quantity ΔS_n^{Ψ} can be computed using the replica method [13,14]. In this context, one evaluates traces of powers of the reduced density matrix ρ_A. This boils down to ratios of expectation values of a twist operator, acting on a replica model composed of n independent copies of the original theory. The operator $\mathbb{T}(A,B)$ acts as a cyclic permutation of the copies $i \mapsto i+1 \bmod n$ on $\mathcal{H}_A^{\otimes n}$, and as the identity on $\mathcal{H}_B^{\otimes n}$, and (2) is expressed as

$$\Delta S_n^{\Psi}(r) = \lim_{\lambda\to\infty} \frac{1}{1-n} \log\left(\frac{{}_n\langle\Psi|\mathbb{T}(\lambda A, \lambda B)|\Psi\rangle_n}{{}_n\langle 0|\mathbb{T}(\lambda A, \lambda B)|0\rangle_n}\right), \qquad (13)$$

where $|0\rangle_n$ is the vacuum state. Both $|0\rangle_n$ and the state $|\Psi\rangle_n$ have the structure

$$|\Psi\rangle_n = |\Psi\rangle^1 \otimes |\Psi\rangle^2 \otimes \cdots \otimes |\Psi\rangle^n. \qquad (14)$$

Here, $|\Psi\rangle^i \simeq |\Psi\rangle$ is the k-particle excited state of interest, implemented in the ith copy.

First, in one dimension, A is a union of segments, and $\mathbb{T}(A,B)$ becomes a product of branch-point twist fields [23] on the boundary points of these segments. Let us consider the case $A = [0,\ell]$ in a periodic system of length L. Then, $\mathbb{T}(A,B) = \mathcal{T}(0)\tilde{\mathcal{T}}(\ell)$, where $\mathcal{T}$ is the branch point twist field and $\tilde{\mathcal{T}}$ is its Hermitian conjugate. In expression (13), one may then expand in a basis $\{|\Phi\rangle\}$ of quasiparticles

$${}_n\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\Psi\rangle_n = \sum_{\Phi} e^{-iP_\Phi \ell} |{}_n\langle\Psi|\mathcal{T}(0)|\Phi\rangle|^2, \qquad (15)$$

where P_Φ are the momentum eigenvalues (in finite volume, they are quantized, and the set of states is discrete). The evaluation of (13) using (15) is, in principle, feasible in integrable $(1+1)$-dimensional QFT, but this presents a number of challenges. Although matrix elements of branch-point twist fields in infinite volume are known [23,43,44], they cannot be used in order to evaluate the limit $L \to \infty$ in (13): divergencies occur whenever momenta of intermediate particles in $|\Phi\rangle$ coincide with those in $|\Psi\rangle_n$. One must first evaluate finite-volume matrix elements, resum the series (15), and take the limit. Finite-volume matrix elements of generic fields are related [45,46] to their infinite-volume counterpart up to exponentially decaying terms in L, but for twist fields, the theory has not been developed yet. We have solved these problems for the massive free real boson and the massive free Majorana fermion. By performing the summation over intermediate states at large L, noting that the so-called "kinematic singularities" of infinite-volume matrix elements provide the leading contribution, we have derived the full results (3) and (4). The details are technical, and presented in a separate paper [47].

Second, we performed a numerical evaluation of the quantity $\Delta S_n^{\Psi}(r)$ using wave functional methods in the harmonic chain and the two-dimensional harmonic lattice, see Fig. 2 and the Supplemental Material (SM) [48]. In the finite-volume Klein-Gordon theory, the vacuum wave functional takes the Gaussian form

$$\langle\varphi|0\rangle \propto \exp\left[-\frac{1}{2}\int_{C\times C} d^d x d^d y K(\vec{x}-\vec{y})\varphi(\vec{x})\varphi(\vec{y})\right], \qquad (16)$$

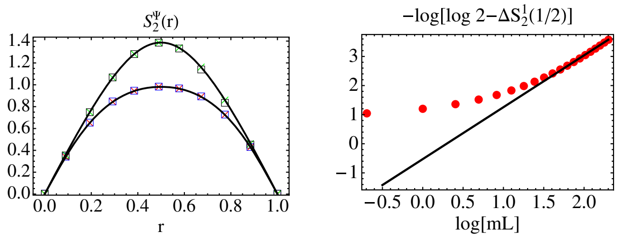

FIG. 2. Numerical results for the case $n=2$ on the toric lattice $[0,L]^2$ with $L=50$ and lattice spacing $\Delta x=1$. Left: Two-particle states $|\Psi\rangle=|\vec{p}_1,\vec{p}_2\rangle$. Squares are for mass $m=1$ and small momenta, crosses are for mass $m=0.001$ and large momenta. The upper curve is formula (4) for $\Delta S_2^{1,1}$, with numerical results for distinct momenta $\vec{p}_1=(0,0)$, $\vec{p}_2=(0.26,0)=(4\pi/L,0)$ (squares) and $\vec{p}_1=(2.51,1.26)=(40\pi/L,20\pi/L)$, $\vec{p}_2=(3.14,0)=(50\pi/L,0)$ (crosses). The lower curve is formula (3) for ΔS_2^2, with numerical results for equal momenta $\vec{p}_1=\vec{p}_2=(0.13,0)=(2\pi/L,0)$ (squares) and $\vec{p}_1=\vec{p}_2=(2.51,1.26)$ (crosses). Right: Approach of $\Delta S_2^1(1/2)$ to the analytical value log 2 for the one-particle state $|\Psi\rangle=|\vec{p}\rangle$ with $\vec{p}=(0,0)$ as a function of L. This shows a linear approach for large mL. The solid line is the fit $0.527-1.783\log(mL)$ on the last eight data points ($mL\in[6.5,10]$).

where $K(\vec{x})=\sum_{\vec{p}}\mathrm{Vol}_d(C)^{-1}E_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}$. Excited state wave functionals have extra polynomial-functional factors, obtained by applying the operator

$$A^\dagger(\vec{p})=\frac{\int_C d^dx e^{i\vec{p}\cdot\vec{x}}(E_{\vec{p}}\varphi(\vec{x})-i\varpi(\vec{x}))}{\sqrt{2E_{\vec{p}}\mathrm{Vol}_d(C)}},\quad [A_{\vec{p}},A^\dagger_{\vec{q}}]=\delta_{\vec{p},\vec{q}},\tag{17}$$

with the representation of the canonical momentum $\varpi(\vec{x})=-i\delta/\delta\varphi(\vec{x})$ satisfying $[\varphi(\vec{x}),\varpi(\vec{y})]=i\delta(\vec{x}-\vec{y})$. Implementing the permutation $\mathbb{T}(A,B)$ on the space of field configurations, the ratio (13) becomes a Gaussian average of polynomial functionals of the fields. With finite lattice spacing Δx, the dispersion relation is $E_{\vec{p}}^2=m^2+4\sum_{i=1}^d\sin^2(p_i\Delta x/2)/(\Delta x)^2$. Numerical results in the one-dimensional case are discussed in more detail in [47], where both QFT and nonuniversal parameter regimes are seen to agree with our predictions, for connected and disconnected regions. Here, we concentrate on the two-dimensional periodic square lattice on $C=[0,L]^2$. We choose a set of subregions $A=[0,\ell]^2$ for values of $r=\ell^2/L^2$ ranging between 0 and 1. In order to establish the validity of the requirements on the correlation length ξ and the de Broglie wavelength ζ, we explore two distinct regimes: that of small ξ but large ζ, and that of small ζ but large ξ, in both cases looking at two-particle states with equal and with distinct rapidities. We find excellent agreement with formulas (4) for $\Delta S_n^{1,1}$ and (3) for ΔS_n^2, respectively, see Fig. 2. Note that the configuration we have chosen is not symmetric: regions A and B have different shapes. Nevertheless, the symmetry $r\mapsto 1-r$ is correctly recovered in the regime of validity of formulas (3) and (4). We have explored other shapes of the region A, obtaining similar accuracy, and have analyzed regimes where both ξ and ζ are small, finding even greater accuracy. We have also analyzed the breaking of formulas (3) and (4) away from their regime of validity. The approach to the maximum log 2 in the case of a single particle with $r=1/2$ (this maximal value is supported by general arguments [34]) is shown in Fig. 2, where the correlation length is varied; we observe an algebraic approach at large mL.

Third, for particular choices of the region A, it is possible to show, analytically, the results (3)–(8) in free models of any higher dimension, by dimensional reduction [49]. Consider the slablike regions $A=[0,\ell]\times C_\perp$ in $C=[0,L]\times C_\perp$ where $C_\perp$ is some $(d-1)$-dimensional space. Construct the canonically normalized one-dimensional Klein-Gordon fields $\varphi(x_1,t)=[\int_{C_\perp}d^{d-1}x_\perp\varphi(x_1,\vec{x}_\perp,t)]/[\sqrt{\mathrm{Vol}_{d-1}(C_\perp)}]$ and similarly for $\varpi(x_1,t)$. This dimensional reduction map preserves the vacuum [49], and the many-particle states when all momenta point in the x_1 direction (with $\vec{p}=(p,0,\ldots,0)$ the expression (17) gives $A^\dagger(\vec{p})=A^\dagger(p)$). Therefore, the quantity ${}_n\langle\Psi|\mathbb{T}(\lambda A,\lambda B)|\Psi\rangle_n$ in d dimensions, is proportional to ${}_n\langle\hat{\Psi}|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\hat{\Psi}\rangle_n$ in 1 dimension. The singularity as $\ell\to 0$ is dimension dependent, but in the ratio (13), this cancels out, and there is exact equality. This analysis extends to other quasi-one-dimensional configurations.

Finally, we establish that our results hold beyond free theories. We analyze the quantity $\Delta S_n^\Psi(r)$ in interacting states of the Bethe ansatz form. Previous analyses exist [33,37], which, however, concentrated on less universal regimes. In the ferromagnetic Heisenberg chain, two-particle states with respect to the ferromagnetic vacuum have the simple form $\sum_{x,y\in\mathbb{Z}}e^{ipx+iqy}S_{\mathrm{sgn}(x-y)}(p,q)|\uparrow\cdots\downarrow_x\cdots\uparrow\cdots\downarrow_y\cdots\uparrow\rangle$, where $S_\epsilon(p,q)$ is the Bethe ansatz scattering matrix. More generally, for the purpose of evaluating large-distance quantities, these are abstract states representing two-particle asymptotic states, with $S_\epsilon(p,q)$ the two-body scattering matrix of the field theory (see the thermodynamic Bethe ansatz formalism of integrable QFT [50,51]). Thus, states of the Bethe ansatz form are expected to provide large-distance results of great generality in integrable models. We have analyzed such one- and two-particle states, and found that formulas (3) and (4) hold, see the SM [48]. There is no need to fix the momenta via the Bethe ansatz; with equal momenta, $S_n^2(r)$ is, indeed, reproduced, extending previous results. Bound states of the Heisenberg chains (Bethe strings) have been studied [33]; these have an intrinsic length scale ξ (inversely proportional to the bounding energy), and one can see that, in the regimes discussed above, $S_n^1(r)$ is, indeed, reproduced. Going beyond integrability, we expect the results to hold, at least, when no particle production occurs, for instance in QFT one-particle states, and two-particle states below the particle production threshold. Any one- and two-particle excitations of Bethe-ansatz form will have EE described by (3)–(8), such as in spin-preserving quantum chains, integrable or not.

Discussion.—It is remarkable that the entanglement of a wide variety of many-body quantum systems admits such a simple and universal "qubit" interpretation. This combines a semiclassical picture of localized particles controlled by correlation lengths and de Broglie wavelengths with the quantum effect of (in)distinguishability. The applicability of Eqs. (3)–(8) to higher dimensions is particularly significant, showing that a large amount of geometric information is irrelevant. Their application to QFT is also interesting: QFT locality is formally based on the vanishing of spacelike commutation relations, not on particles, yet our results show how quantum entanglement clearly "sees" localized particles. This suggests that entanglement entropy could be used as a diagnostic tool for determining whether excitations are of a quasiparticle type. The relation (12) suggests that quasiparticle excitations in extended systems of any dimension can be used to create simple entangled states with controllable entanglement, where the control parameter is the region-to-system volume ratio r. It would be interesting to investigate the possible applications of such a result in the area of quantum information.

We are grateful to EPSRC for funding through the standard proposal "Entanglement Measures, Twist Fields, and Partition Functions in Quantum Field Theory" under Grants No. EP/P006108/1 and No. EP/P006132/1. We would also like to thank Vincenzo Alba for discussions and for bringing Ref. [33] to our attention.

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